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OPTIMALITY CONDITIONS FOR FINITE ELEMENT SIMULATION OF ADAPTIVE BONE REMODELING

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Abstract—Bone remodeling in vertebrates is widely quoted as a process which optimizes the use of structural material, subject to mechanical requirements. In vertebrates, bone remodeling continues throughout life and tends to preserve the structure of a particular bone over decades of life. This implies that the processes which remodel bone are stable, at least over a time period spanning many years. Many recent numerical simulations of bone remodeling have used rate equations which have not been carefully assessed for stability and their ability to produce an optimal structure.

In this study, we re-state the conditions necessary for stability of a particular bone remodeling rate equation derived in a related study, and we investigate whether the rate equation used can produce an optimal structure. Within the context of a finite element discretization, we show that this rate equation does not produce a structure optimized with respect to density. By making a simple modification to the stable remodeling rate equation, we show that the remodeling stimulus used can produce an optimal structure if the state variable manipulated is density taken to a power.

We conclude that if bone is a stable, self-optimizing structure, there are specific requirements for the point-by-point rate of change of bone density in response to mechanical stress. The implications of these requirements for simulations of adaptive bone remodeling are discussed.

INTRODUCTION

The structure of bones within the skeletal system of vertebrates has often been cited as an example of a natural system in which the adaptation of a tissue to mechanical requirements has been achieved. Bone remodeling is a general term which describes the processes which maintain bone throughout adult life. Tissue is laid down and resorbed at surfaces within the bone microstructure, since the rigid nature of the extracellular matrix prevents growth within the volume of the tissue. The biological principals which govern bone remodeling are currently under a great deal of debate, and numerical simulations of bone remodeling show that the influence of mechanical effects in remodeling are also difficult to assess. The ability of these simulations to correctly mimic the remodeling processes in bone is still being assessed, and there is no remodeling rate equation which has been shown to be definitively better than others. Although bone is constantly being laid down and removed in order to compensate for fatigue damage, no current theories for remodeling can account for this base-line cellular activity. The term "bone remodeling" is usually used to describe changes in the shape of bone and changes in the distribution of bone density with time. Thus, by continually maintaining and adapting the structure of bones, the biological processes which remove fatigued material and lay down new material within bone illustrate the degree of mechanical sophistication which can occur in nature.

Wolff's law is the most widely known theory for bone adaptation to stress. Wolff's law is an extrapolation made by observing the correlation between patterns seen in a section through the cancellous bone in a number of joints and the lines drawn in an analysis of stress using the method of characteristics [see Wolff (1986)]. The theory is a qualitative one which states that the internal and external structure of bone changes to adapt to changes in the loads applied to it. Wolff's law is often interpreted today as a postulate that bone is distributed and oriented so as to support the loads placed on it with the minimum amount of material. This material optimization theory is very widely accepted as a simple explanation for a complex bony structure, but a rigorous proof of the principal has not been arrived at. A number of numerical models have been exploited to study specific characterizations of Wolff's law recently. These studies can qualitatively predict the density

distribution of bone near joints, and the shape of long bone diaphyses. Thus the original qualitative postulates made by the early anatomists have been confirmed, and a number of specific implementations of this postulate are being assessed at this time. Although these structural studies use continuum-averaged quantities (strain energy density and stress measures), and do not assess the details of the remodeling processes per se, they provide an indication of the mechanical factors which are important within the microstructure.

In this study, we are using continuum-level remodeling rate equations to assess two qualitative questions regarding these models. First, are these simulations stable? Second, can certain remodeling rate postulates lead to an optimal structure, as proposed by certain interpretations of Wolff's law? We have identified some requirements on a remodeling rate equation for a structure which satisfies these two requirements. We believe these requirements can serve as checks on specific biologically-oriented remodeling rules for bone.

Bone is generally classified into two types—cortical bone, the dense layer of material on the outside of bones, and cancellous bone, the porous material found around joints. Cortical bone has low porosity, and an elastic modulus which is reasonably well known (Reilly and Burstein, 1974). Cancellous bone is highly porous, and has an elastic modulus which is very variable. The elastic modulus of cortical bone varies with porosity to some extent, and the elastic modulus of cancellous bone depends heavily on the tissue porosity and orientation (Carter and Hayes, 1977; Rice *et al.*, 1988). Some authors consider cortical bone and cancellous bone together, with a power law relationship between density and elastic modulus throughout the range of density (Carter and Hayes, 1977). In order to keep the formulation tractable, we have idealized cancellous bone as isotropic, with a constant Poisson ratio and an elastic modulus which is related to the density of the tissue. We are concentrating this study on remodeling in cancellous bone, and we have not assessed the effects of geometric changes on the remodelling response.

The cells which remodel bone seem to act on a local basis within the tissue, responding to local cues from the micro-environment. That is, no anatomic or physiological information is available which suggests that a small region of bone is able to sense its anatomic position and act according to that information (Netter, 1987; Warwick and Williams, 1973). Thus the information available to the cells which remodel bone seems to be primarily the local tissue stress and the available biochemical information (such as oxygen, hormonal or nutrient concentrations). This indicates that the overall structure of a bone is produced by the concerted action of remodeling processes which act microscopically. Clearly, there are significant non-mechanical effects, as in the skull (lightly loaded, yet containing dense bone), but in this paper we will restrict attention to the mechanical factors. This restricts the application of this research to regions where non-mechanical effects are either small or are equally applied to the entire region being modeled.

The form of the remodeling rules proposed for bone thus far is fairly simple: the rate of change of bone density at a specific point is proportional to a "stimulus" which is related to the stress state of the bone and the bone density at that point. For example, some investigators (Huiskes *et al.*, 1987; Orr *et al.*, 1989) use a stimulus which is strain energy density or a measure of stress, divided by bone density, and subtracted from a "null state". This "null state" is a value of stimulus at which the rate of change of density is zero. This remodeling stimulus can be taken as a constant value throughout the structure, or as a position-dependent stimulus. A position-dependent stimulus is possible, due to non-mechanical effects. An adjustable "null state" must be carefully used, however, since the liberal use of an adjustable parameter such as this can render the analysis meaningless.

Careful study of the processes used to simulate remodeling in these studies offers an opportunity to make the qualitative postulates of the early anatomists more specific. The temporal stability of the numerical bone remodeling simulations has been discussed in a number of recent studies. Carter *et al.* (1989) discussed convergence issues, as did Fyhrie and Hollister (1990). Weinans *et al.* (1989, 1990), Weinans (1991) and Huiskes *et al.* (1991) have explored the stability of bone remodeling simulations directly. Using a model with two discrete remodeling elements, Weinans *et al.* (1990) indicated that the origin of the unstable behavior was the remodeling rule, rather than the details of the finite element approximations. Huiskes *et al.* (1991) have shown that the density distribution in bone can

be simulated, if the strain energy density from an intact bone is used as the position-dependent "attractor state" in a fairly coarse finite element mesh. The stability of this model was considered tenuous, and refinement of the finite element mesh was not attempted. Harrigan and Hamilton (1992a) have indicated from a continuous analytical model that the stability of bone remodeling simulations depends on the form of the remodeling rule. The analytical model resulted in the derivation of a stability limit for mechanical bone remodeling postulates, and showed that the unstable behavior which resulted from failure to satisfy the limit was not similar to the behavior seen *in vivo*. Harrigan and Hamilton (1992b) have developed a stability analysis for bone remodeling using finite element techniques, and have shown that the requirement for stability as derived in the previous analytical model is directly reflected in a finite element model. The stability of the algorithms used to simulate the temporal evolution of the density distribution were also studied, leading to an effective Euler backward time-stepping procedure. This study builds on the previous finite element work, and develops ways in which the remodeling process developed can be considered as an optimal process. By optimal, we mean a process which manipulates state variables within the structure in order to minimize a function of those state variables.

ANALYSIS

In order to assess the ways in which the remodeling rule used here can be regarded as an optimizing process, we will first review the development of the stability conditions using finite element analysis. The conditions for an optimal structure which are derived here are based on a particular stimulus for bone remodeling, using strain energy density. Other bone remodeling theories, based on stress and strain measures, have been developed. We chose to concentrate on a remodeling stimulus based on strain energy density, since it is mathematically simpler than stress measures.

Stability limit based on finite element discretization

An incremental analysis has been developed in order to study the reaction of the structure being investigated to small perturbations in density. For the purposes of the stability analysis, the structure being analysed is assumed to be at a state of remodeling equilibrium : i.e. we assume the density distribution has evolved to a steady state situation. Stability conditions in this analysis are developed using a standard method for study of a dynamic system, with the eventual tests for stability carried out using numerical methods.

In this development, and the development to follow regarding optimality conditions, the element matrices are written in global coordinates. The two indices are assumed to span the range of global coordinate indices, with non-zero entries only in the positions in which the global coordinates couple to the element. Thus we never refer to local coordinates for the element stiffness matrices we use here. Also, the implied summation over repeated indices is only used when the repeated indices are subscripts which follow displacements or stiffness matrices. When summation over elements is implied, it is explicitly expressed using a summation sign. We will also use the subscripts f, g, p and q to denote particular (global) degrees of freedom. The subscripts and left superscripts e and s refer to element number. When a quantity is a right superscript (used here only on density), exponentiation is meant. Thus we have restricted our notation in order to maintain clarity in the operations we intend.

We assume a material Poisson ratio which is independent of density, and a power law relationship between density, ϕ, and elastic modulus (Carter and Hayes, 1977), i.e.

$$E = E_0\phi^n. \tag{1}$$

This results in a material property matrix $[C]$ which can be expressed as

$$[C] = [C_0]\phi^{(n)}, \tag{2}$$

with ϕ the volumetric density, n the material exponent, and $[C_0]$ the material property

matrix for unit volumetric density (Bathe, 1982). Using this matrix in a standard finite element formulation for element stiffness matrices yields

$$^{e}L_{fg} = \phi^{(n)}\,{}^{e}K_{fg}, \tag{3}$$

with the superscript e denoting which element is used, $^{e}L_{fg}$ the element stiffness matrix and $^{e}K_{fg}$ the element stiffness matrix for unit density.

The average strain energy density within the element, $^{e}\Psi$, for a given displacement solution u is

$$^{e}\Psi = \frac{\phi^{(n)} u_f\,{}^{e}K_{fg} u_g}{2V_e}, \tag{4}$$

with V_e the element volume. We assume the tissue stimulus for bone remodeling is

$$^{e}\Psi_t = \frac{^{e}\Psi}{\phi^{(m)}}, \tag{5}$$

with the exponent m a material constant. This equation is meant to reflect the fact that the continuum-level stresses, strains and strain energy density are typically magnified within the microstructure, so that the stresses, strains and strain energy density within the microstructure are higher than the continuum-level variables. Beaupré *et al.* (1990) proposed a stimulus which accounts for the fact that the strength of cancellous bone is proportional to the square of the density of the tissue, so that the stress within the tissue is related to the overall stress in the inverse fashion. Since we are using strain energy density in this formulation, and since we are not making specific assumptions about the specific physical stimulus for remodeling, we chose this relationship with the exponent m left as a parameter in the relationship.

Using this stimulus, the overall remodeling rate equation is

$$\frac{\partial \phi_e}{\partial t} = {}^{e}\Psi_t - \Psi_0. \tag{6}$$

Here the symbol Ψ_0 denotes a "null point", where the remodeling rate vanishes. In general, there is a rate coefficient multiplying the stimulus (the right-hand side of the equation), and for simplicity we have chosen to scale time so that the rate coefficient is unity. The overall rate equation can thus be written as

$$\frac{\partial \phi_e}{\partial t} = \frac{\phi_e^{(n-m)} u_f\,{}^{e}K_{fg} u_g}{2V_e} - \Psi_0. \tag{7}$$

The finite element equations to be solved are $\mathbf{K}_{pq}u_p = R_q$, with $\mathbf{K}_{pq}$ the overall stiffness matrix, formed by a sum of the element stiffness matrices, and R_q the load vector for the finite element mesh. Taking perturbations of the finite element equations gives

$$\mathbf{K}_{pq}\Delta u_p = -\Delta \mathbf{K}_{pq} u_p, \tag{8}$$

with $\Delta \mathbf{K}_{pq}$ the change in stiffness for a perturbation in density. This matrix can be written, with $\Delta\phi_s$ the change in density, as

$$\Delta \mathbf{K}_{pq} = \sum_{s=1}^{N} n\phi_s^{(n-1)}\,{}^{s}K_{pq}\Delta\phi_s. \tag{9}$$

This equation comes from taking derivitives of the total stiffness matrix, written as a sum

of individual element stiffness matrices, with respect to density in each element. Thus, the change in displacements for a change in density distribution can be written as

$$\Delta u_p = -K_{pq}^{-1} \sum_{s=1}^{N} n\phi_s^{(n-1)} \Delta\phi_s {}^{s}K_{qq} u_q. \tag{10}$$

Perturbations in the rate equation, assuming a converged state, can be written as

$$\frac{\partial\phi_e}{\partial t} + \frac{\partial(\Delta\phi_e)}{\partial t} = \frac{(\phi_e^{(n-m)} + (n-m)\phi_e^{(n-m-1)}\Delta\phi_e)(u_f {}^{e}K_{fg} u_g + 2u_f {}^{e}K_{fg}\Delta u_g + \Delta u_f {}^{e}K_{fg}\Delta u_g)}{2V_e} - \Psi_0, \tag{11}$$

where we have used the fact that the element stiffness matrices are symmetric. Canceling terms, neglecting higher order terms in perturbations, and substituting for Δu gives

$$\frac{\partial(\Delta\phi_e)}{\partial t} = M_{es}\Delta\phi_s + D_{es}\Delta\phi_s, \tag{12}$$

with

$$M_{es} = \frac{-n\phi_e^{(n-m)}\phi_s^{(n-1)} u_f {}^{e}K_{fp} K_{pq}^{-1} {}^{s}K_{qq} u_q}{V_e}, \tag{13}$$

and

$$D_{es} = \delta_{es} \frac{(n-m)u_f {}^{e}K_{fg} u_g}{2V_e} \phi_e^{(n-m-1)}, \tag{14}$$

with δ_{es} equal to 1 if $e = s$, and zero otherwise.

Based on a solution which is exponential in time, the solution of eqn (12) will decay with time, and the equilibrium state arrived at will be stable, if the matrix $M_{es} + D_{es}$ is negative definite. A matrix is negative definite if all the eigenvalues of the matrix have negative real parts. Since the matrix derived here is non-symmetric, the eigenvalues will be complex numbers, and special techniques for eigenvalue extraction are needed to compute them. Also, since the equations in perturbations are linear, the matrix D_{es} can be considered separately. This shows that the analytical stability requirement (Harrigan and Hamilton, 1992a) is preserved. The analytical requirement that n be less than m is necessary to keep the coefficients in this diagonal matrix less than zero.

Harrigan and Hamilton (1992b) have shown a method whereby the stability of the remodeling simulation can be assessed without actually computing the eigenvalues of this non-symmetric matrix. The procedure consists of factorizing the symmetric part of the matrix into an LDL^{T} form and assessing whether the diagonal matrix D contains negative elements (Crandall, 1955; Lancaster and Tismenetsky, 1985; Strang, 1986).

Study of optimality conditions in strain energy-based bone remodeling theories

An optimal structures approach to bone is useful for theoretical and practical reasons. The proposition that bone is an optimal structure is one of the cornerstones of medical thought regarding bone. This proposition fits closely with other biological theories regarding natural selection for advantageous traits. If simulations of the natural remodeling situation which closely mimic *in vivo* behavior are found to optimize a given quantity relative to a certain state variable, the theoretical framework for these optimization postulates will be strengthened. Also, if we can identify a particular state variable as being manipulated to optimize the structures studied, we can indicate ways to frame or interpret studies of the

physical process within the microstructure. Practically, a formulation for bone remodeling which optimizes a function relative to a state variable is far easier to use than one which does not, since the large literature on optimization can be used to good effect. Both the practical and the theoretical issues regarding bone remodeling are important, since the conceptual issues are much debated, and simulations of bone remodeling often encounter numerical problems.

In order to assess whether these simulations result in a structure which minimizes some functional on density, one can make an analogy to the weak solution of differential equations. For example, Hamilton's principle as applied to elasticity problems shows that by minimizing an energy functional one can solve the elasticity field equations. Thus solving the elasticity equations corresponds to finding a displacement distribution which minimizes a functional (stored elastic energy). In this study, we make an assessment of whether the density distribution minimizes a functional on density if the density distribution is given by a long-term (i.e. "steady state") solution of the remodeling rate equation. That is, we have assessed whether a density distribution which minimizes a strain energy based "remodeling stimulus" can be considered to minimize a functional on density.

If a functional exists which is minimized at steady state by the remodeling rule, then by analogy to Hamilton's principal, a solution can be found by minimizing the functional with respect to arbitrary admissible variations in density. Also, the second variation in this functional should be symmetric. That is, the first variation in the functional will yield the equilibrium equations, and the second variation in the functional (the first variation in the equilibrium equations) should be symmetric with respect to the two variations used.

In a discretized version of the remodeling problem, the ideas can be made more concrete. If there is an indicator function which measures some property of the density distribution, and results in a specific numerical value, then we can write that function as

$$H(\phi_1, \phi_2, \phi_3, \ldots, \phi_n) \tag{15}$$

and if the remodeling rule used minimizes (or finds stable points of) this indicator function, then the remodeling rule can be given by

$$F(\phi_i) = \frac{\partial H(\phi_1, \ldots, \phi_n)}{\partial \phi_i}. \tag{16}$$

Thus we can show that

$$\frac{\partial F(\phi_i)}{\partial \phi_j} = \frac{\partial^2 H(\phi_1, \ldots, \phi_n)}{\partial \phi_i \partial \phi_j} = \frac{\partial^2 H(\phi_1, \ldots, \phi_n)}{\partial \phi_j \partial \phi_i} = \frac{\partial F(\phi_j)}{\partial \phi_i}. \tag{17}$$

Thus, a remodeling rule can be derived from an overall indication function on density if and only if the derivatives of that rule follow this equation.

In our stability studies, we have used a particular remodeling rule in which

$$\frac{\partial \phi_i}{\partial t} = F(\phi_i), \tag{18}$$

and we have calculated the derivatives in eqn (17) for that rule, arriving at the matrices M_{rs} and D_{rs}. Since M_{rs} is not equal to M_{sr}, the derivatives are not equal for that particular rule. Thus, the mathematical form of that role is not the result of optimizing an indicator function on density.

Careful study of eqn (13) shows that one possible way in which the matrix M_{rs} can be taken as symmetric would be to take $m = 1$. This would indeed satisfy the conditions for optimality, but as was shown in our previous studies (Harrigan and Hamilton, 1992a, b), this choice would result in an unstable remodeling stimulus. Thus, it seems as though this

choice for remodeling stimulus can produce either a stable structure or an optimal structure, but not both.

There is a way to arrive at a simulation of bone remodeling which uses the above rule and which satisfies the mixed partial derivative condition, however. It involves modifying the rate equation in such a way as to change the variable which is manipulated to optimize the function.

Consider the following changes in the remodeling rate equation. Instead of using density as our state variable, we can choose density to some positive power, as in

$$\gamma = \phi^z, \tag{19}$$

with z a positive exponent, which we will arrive at later in this discussion. Note that when γ is equal to zero, ϕ is equal to zero, and when γ is equal to one, ϕ is equal to one. If we re-cast the rate equation as follows:

$$V_e \frac{\partial \gamma_e}{\partial t} = \frac{\gamma_e^{(n-m)/z}(u_f^T K_{fa} u_a)}{2} - \Psi_0 V_e, \tag{20}$$

we will have the same equilibrium states as before, since

$$\frac{\partial \phi_e}{\partial t} = 0 \Leftrightarrow V_e \frac{\partial \gamma_e}{\partial t} = 0 \quad (\phi_e \neq 0), \tag{21}$$

and the right-hand side of eqn (20) is just the right-hand side of eqn (7), multiplied by each element volume.

The situation when ϕ_e approaches zero needs consideration, since at that point, the derivative of γ_e will be zero even if the derivative of ϕ_e is not. From physical considerations, this seems like a strength, rather than a weakness, since the volumetric density in a particular region cannot be less than zero. In general, if the volumetric density approaches zero in a steady-state distribution, the matrix product in eqn (20) also approaches zero. Since $n-m$ is negative for stability, both ϕ_e^{n-m} and $\gamma_e^{(n-m)/z}$ will approach infinity. In a region where density goes to zero, these two factors approach zero individually, and they do so in such a way that their product is equal to Ψ_0.

This remodeling rule will have different temporal behavior, but the same equilibrium states as eqn (7), as shown below. Since no current bone remodeling rule purports to closely mimic transient remodeling processes, we do not consider our modification to be a serious problem for the theoretical development.

The left-hand side of eqn (20) could be troubling, since element volume appears there. We have manipulated element volume here to remove it from the first term on the right-hand side of eqn (20). Since the equilibrium state for bone remodeling has the right-hand side of eqn (20) as zero within every element of the finite element model, multiplying the right-hand side of eqn (20) by element volume will not alter the equilibrium states. Also, since the stimulus is calculated as the actual stimulus multiplied by element volume, and the rate calculation divides this stimulus by the same element volume, the effect of the finite element mesh parameters will not influence the rate calculation any more than would be expected from a typical dynamic finite element model. Thus, the change in temporal behavior of the model between eqns (7) and (20) is related to calculating the the rate of change of γ, instead of the rate of change of ϕ.

Although this remodeling rule seems artificial, it can be shown to be the result of minimizing a functional on γ if the exponent z is chosen equal to m. Since the remodeling stimulus is the same in this equation as in the other equations shown, and since γ is directly related to ϕ, we believe this result can have important implications for bone remodeling.

In order to show that choosing $z = m$ will satisfy the mixed partial derivative conditions, we must first assess the changes in displacements due to changes in density. The

finite element equations for equilibrium can be written as $KU = R$, and if we expand the global matrix K, we can write

$$K_{fg} = \sum_{e=1}^{N} \phi_e^n {}^eK_{fg} + {}^cK_{fg}, \tag{22}$$

or, written in terms of γ,

$$K_{fg} = \sum_{e=1}^{N} \gamma_e^{n-z} {}^eK_{fg} + {}^cK_{fg}, \tag{23}$$

with ${}^cK_{fg}$ the part of the stiffness matrix which remains constant during the remodeling simulation (e.g. an orthopedic implant). Thus we can take the derivative of $KU = R$ with respect to γ_e and arrive at

$$\frac{\partial}{\partial \gamma_e}(u_f) = -(K_{fp}^{-1} {}^eK_{pq} u_q) \frac{n}{z} \gamma_e^{(n-z)/z}. \tag{24}$$

In order to choose z properly, we write the rate equation as

$$V_e \frac{\partial \gamma_e}{\partial t} = F(\gamma_e), \tag{25}$$

and we take derivatives as

$$\frac{\partial F(\gamma_e)}{\partial \gamma_s} = \delta_{es} \binom{n-m}{z} \gamma_s^{[(n-m-z)/z]} \frac{(u_f {}^sK_{fg} u_g)}{2} + 2\gamma_e^{[(n-m)/z]} \frac{u_f {}^eK_{fg}}{2} \frac{\partial u_g}{\partial \gamma_s}. \tag{26}$$

Substituting from eqn (24), we have

$$\frac{\partial F(\gamma_e)}{\partial \gamma_s} = \delta_{es} \binom{n-m}{z} \gamma_s^{[(n-m-z)/z]} \frac{(u_f {}^sK_{fg} u_g)}{2} - \gamma_e^{[(n-m)/z]} (u_f {}^eK_{fp} K_{pq}^{-1} {}^sK_{qq} u_q) \binom{n}{z} \gamma_s^{[(n-z)/z]}. \tag{27}$$

The term δ_{es} in this equation is equal to 1 if $e = s$ and zero otherwise, so the first term of the right of this equation is symmetric in e and s by definition. The second term on the right-hand side of this equation contains a rather complicated matrix product which can be shown to be symmetric in the indices e and s, since K_{pq}^{-1} is the inverse of the symmetric global finite element matrix. In order to make the exponents on the terms γ_e and γ_s equal, we need only choose $z = m$. This choice will make eqn (27) symmetric with respect to indices e and s, and it indicates that eqn (17) is satisfied for this rate equation if we take derivatives with respect to γ instead of ϕ.

DISCUSSION

By studying the mathematical character of the remodeling rate equations, we have been able to assess two issues regarding bone remodeling as characterized by a remodeling rule based on strain energy density: stability and optimality. In a related study (Harrigan and Hamilton, 1992b), we used a finite element discretization to show an interrelationship between the bone remodeling process on a microscopic scale and the stability of the overall structure of bone on a global scale. This study confirmed stability requirements derived in a previous analytical study [Harrigan and Hamilton (1992a), and see after eqn (14)]. We also arrived at a method to simulate temporal changes in bone density distributions within a finite element analysis. In this study, we build on the finite element study of remodeling

stability to study conditions for the presence of an optimal structure in bone. More specifically, we have arrived at a set of state variables, derived from a specific bone remodeling postulate, which can be considered to extremize a function of these variables over the entire structure. Although we expect that the arguments put forward here do not depend on the finite element discretization, we have kept our development within this framework for simplicity in notation. A more general statement of the problem and results using functional analysis is probably possible, but is not attempted here.

From this analysis of optimal conditions for bone remodeling we have arrived at a choice for the remodeling state variable which makes the remodeling rule part of an optimizing procedure. A derivation of the actual functional which is minimized has not been accomplished, but an algorithm to test candidate bone remodeling rules for stability and optimizing behavior has been arrived at.

The function which the remodeling rate equation minimizes can theoretically be derived by integrating the rate equation appropriately, but we have been unable to do this as yet. We have found a method to differentiate the complicated dependence of the displacements on the density distribution, but we have not been able to integrate the relationship in a sensible fashion as yet.

The specific form of the rate equation in γ which will result in an optimal structure can be related back to the rate equations in ϕ by substituting for ϕ^z for γ and by expanding the derivative with respect to time. This results in

$$m\phi_e^{m-1}\frac{\partial \phi_e}{\partial t} = \frac{\phi_e^n u_f^e K_{fg} u_g}{2\phi_e^m V_e} - \Psi_0, \tag{28}$$

which can be written as

$$\frac{\partial \phi_e}{\partial t} = \frac{\phi_e^n u_f^e K_{fg} \mu_g}{2m\phi_e^{(2m-1)} V_e} - \frac{\Psi_0}{m\phi_e^{(m-1)}}. \tag{29}$$

The form of this equation can be seen as very close to the form of eqn (7), and this equation is also very close to those studied in the literature (Huiskes *et al.*, 1987, 1991; Weinans, 1991; Weinans *et al.*, 1989, 1990). Since taking the time derivative equal to zero for each element allows one to factor out m and $\phi^{(m-1)}$, the stable long-term states for bone remodeling will be the same in this analysis as with a strain energy density formulation such as in Harrigan and Hamilton (1991a). The complication as ϕ_e becomes small, which was discussed after eqn (21), is still evident, only in eqn (29), the right-hand side is further divided by $\phi^{(m-1)}$, so that the conditions necessary for the simulation to approach equilibrium for regions with very low densities are somewhat stricter than in eqn (7). The requirement for equilibrium is now that the entire right-hand side of eqn (20) approaches zero (with time) faster than $\phi_e^{(m-1)}$. However, this requirement can be eliminated by simply using eqn (20).

CONCLUSIONS

The optimizing nature of bone remodeling is important from a theoretical and practical standpoint. Theoretically, bone has been postulated as an optimal structure for over a century, and much of the biological and medical knowledge of bone adaptation is interpreted relative to this assumption. Practically, an optimization procedure is much easier to apply than a point-by-point time stepping procedure.

In a previous study (Harrigan and Hamilton, 1992a), we have shown a modification to existing bone remodeling algorithms which is necessary for stable time-stepping of these equations. In this study, we show that by modifying the state variable used to characterize the temporal response of bone to strain energy density, we can produce a simulation which satisfies the conditions for the existence of an overall cost function which is extremized.

We have derived a stable simulation of bone remodeling which is optimal in some sense, and which is similar to those in the literature which appear to correctly mimic bone

density distributions in a qualitative fashion. These simulations satisfy two qualitative mathematical requirements (stability and optimality) and are similar to simulations which qualitatively mimic bone density distributions. Further numerical analyses will be needed to assess whether the modified simulations we arrive at here will correctly mimic the natural bone remodelling in a quantitative sense.

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